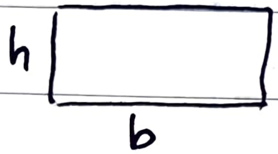
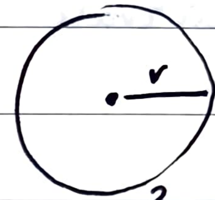
Lec 07

No. Date. July 9, 2026



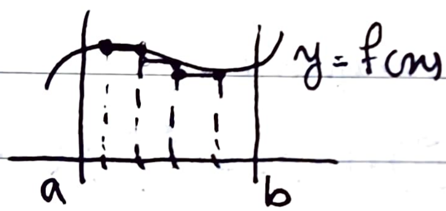
$A = bh$

$A = \frac{1}{2}bh$

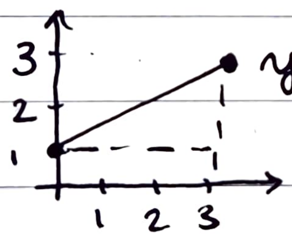


$A = \pi r^2$

How do we define the area of a random region?



The sum of the areas of the rectangles = Area



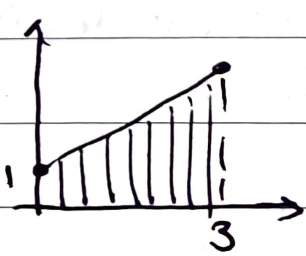
$y = \frac{2}{3}x + 1$

$A = \triangle + \square$

$(\frac{1}{2} 3 \cdot 2) + (3 \cdot 1) = \boxed{6\ m^2}$

Right hand rule:

use equal-width subintervals & right-hand endpoints for the heights of the rectangles.



n equal pieces or subintervals of width $\frac{3-0}{n}$

No. Date.

subintervals | right-hand endpoints

subintervals	right-hand endpoints
1	$0+1\left(\frac{3-0}{n}\right)$
2	$0+2\left(\frac{3-0}{n}\right)$
3	$0+3\left(\frac{3-0}{n}\right)$
$\vdots$	
n	$0+n\left(\frac{3-0}{n}\right)=3$

Sum of the areas of the rectangles:

$$\left(\frac{3-0}{n}\right)\cdot f\left(0+1\cdot\frac{3-0}{n}\right)+\left(\frac{3-0}{n}\right)\cdot f\left(0+2\cdot\frac{3-0}{n}\right)+\cdots$$

$$+\left(\frac{3-0}{n}\right)+\left(0+n\cdot\frac{3-0}{n}\right)=$$

$$\sum_{i=1}^{n}\left(\frac{3-0}{n}\right)\underbrace{f\left(0+i\frac{3-0}{n}\right)}_{n}=$$

$$\sum_{i=1}^{n}\left(\frac{3-0}{n}\right)\left[\frac{2}{3}\left(0+i\frac{3-0}{n}\right)+1\right]$$

Simplify $\Rightarrow \displaystyle\sum_{i=1}^{n}\frac{3}{n}\left(\frac{2}{3}\cdot\frac{3i}{n}+1\right)\Rightarrow$

$$\frac{3}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}+1\right)$$

So the Area under the function:

$$\text{Area}=\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}+1\right)$$

No. Date.

$$\lim_{n\to\infty} \frac{3}{n}\left[\left(\sum_{i=1}^{n} \frac{2i}{n}\right) + \left(\sum_{i=1}^{n} 1\right)\right] =$$

$$\lim_{n\to\infty} \frac{3}{n}\left[\frac{2}{n}\left(\sum_{i=1}^{n} i\right) + n\right] =$$

★ we know:

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+(n-1)+n$$

& based on that we can write:

$$\begin{array}{cccccccccccccc} & 1 & + & 2 & + & 3 & + & \cdots & + & (n-2) & + & (n-1) & + & n \\ + & n & + & (n-1) & + & (n-2) & + & \cdots & + & 3 & + & 2 & + & 1 \end{array}$$

$$= (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1) =$$

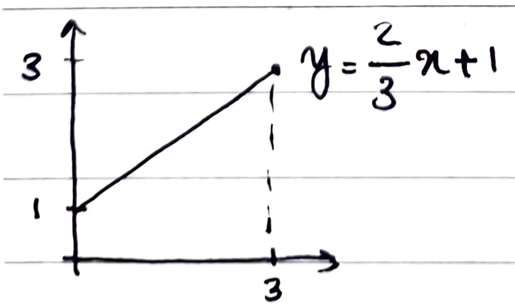
$$n(n+1)$$

therefore:

$$\lim_{n\to\infty} \frac{3}{n}\left[\frac{2}{n}\left(\frac{n(n+1)}{2}\right) + n\right] \Rightarrow$$

$$\lim_{n\to\infty} \frac{3}{n}\left[2n+1\right] = \lim_{n\to\infty}\left[\frac{(3)(2)n}{n} + \frac{3}{n}\right] = \lim_{n\to\infty}\left[6+\frac{3}{n}\right]$$

$$= 6 + 0 = \boxed{6}\, m^2$$



How do we really do this with calculus?

No. Date.

what function has derivative $\frac{2}{3}x + 1$?

$$\frac{2}{3}\left(\frac{x^2}{2}\right) + x \Rightarrow \frac{x^2}{3} + x \Big|_0^3 \Rightarrow$$

$$\left(\frac{3^2}{3} + 3\right) - \left(\frac{0^2}{3} + 0\right) = 3 + 3 = \boxed{6}\ m^2$$

$$\int_0^3 \left(\frac{2x}{3} + 1\right) dx$$

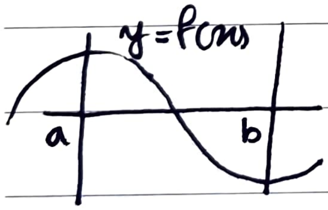
Fundamental Thm of calculus:

I. if $F'(x) = f(x)$, (& $f(x)$ is integrable) then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

II. if $F(x) = \int_c^x f(t)\,dt$ then $F'(x) = f(x)$

c (any constant)



$\int_a^b f(x)\,dx$ represents weighted area A − B

*Area below the x-axis is negative.

Rules for integration (anti derivatives) vs Particular Integrals

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ a definite integral

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$ an indefinite integral

Rules:

$$\int x^n . dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & n \neq -1 \\ \ln(x) + C & n = -1 \end{cases}$$

$$\int c f(x) . dx = c \int f(x) . dx$$

$$\int (f(x) + g(x)) . dx = \int f(x) . dx + \int g(x) . dx$$

$$\int_a^b f(x) . dx + \int_b^c f(x) . dx = \int_a^c f(x) . dx$$

$$\int_a^a f(x) . dx = 0$$

$$\int_a^b f(x) . dx = - \int_b^a f(x) . dx$$

No. Date.

$$\int_{19}^{42} (2x^2 - 3)\,dx = 2\int_{19}^{42} x^2 . dx - 3\int_{19}^{42} 1 . dx \Rightarrow$$

$$2 . \frac{x^3}{3}\Big|_{19}^{42} - 3 . x\Big|_{19}^{42} \Rightarrow$$

$$\left(\frac{2x^3}{3} - 3x\right)\Big|_{19}^{42} \Rightarrow$$

$$\left(\frac{2}{3} 42^3 - 3(42)\right) - \left(\frac{2}{3} . 19^3 - 3(19)\right)$$

More rules:

$$\int e^x . dx = e^x + c$$

$$\int \cos(x) . dx = \sin(x) + c$$

$$\int \sin(x) . dx = -\cos(x) + c$$

$$\int \tan(x) . dx = \int \frac{\sin(x)}{\cos(x)} . dx = -\ln(\cos(x)) + c$$

$$\int \sec(x) . dx = \ln(\sec(x) + \tan(x)) + c$$

No. Date.

Even more rules:

$$\int \ln(x)\cdot dx = x\ln(x) - x + C$$

$$\Rightarrow \frac{d}{dx}(x\ln(x) - x) = (1)\ln(x) + x\left(\frac{1}{x}\right) - 1 = \ln(x)$$

The (basic) substitution rule (or the chain rule reversed).

$$\int f'(g(x))\,g'(x)\cdot dx = f(g(x)) + C$$

$$\Rightarrow \int f(g(x))\,g'(x)\cdot dx = \int f(u)\,du$$

$u = g(x) \quad du = g'(x)dx$

example:

$$\int \tan(x)\cdot dx = \int \frac{\sin(x)}{\cos(x)}\cdot dx = \int \frac{1}{\cos(x)}\cdot \sin(x)\cdot dx \Rightarrow$$

$u = \cos(x) \quad du = -\sin(x)dx \Rightarrow$

$(-1)du = \sin(x)dx$

$$\Rightarrow \int \frac{1}{u}(-1)du = -\ln(u) + C = \boxed{-\ln(\cos(x)) + C}$$

example:

$$\int \frac{x^2}{x^3+4}\cdot dx = \int \frac{1}{w}\left(\frac{1}{3}dw\right) = \frac{1}{3}\ln(w) + C \Rightarrow$$

$$\boxed{\frac{1}{3}\ln(x^3+4) + C}$$

$w = x^3 + 4$

$dw = 3x^2\cdot dx \Rightarrow$

$\frac{1}{3}dw = x^2 dx$

No. Date.

example :

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\cdot dx$$

$u = \sin(x)$

$du = \cos(x)\cdot dx$

$$= \int_{x=0}^{x=\pi/2} u^2 du = \frac{u^3}{3}\Bigg|_{x=0}^{x=\pi/2} = \frac{\sin^3(x)}{3}\Bigg|_0^{\pi/2} \Rightarrow$$

$$\frac{\sin^3(\frac{\pi}{2})}{3} - \frac{\sin^3(0)}{3} = \frac{1^3}{3} - \frac{0^3}{3} = \boxed{\frac{1}{3}}$$

x	u
0	0
$\pi/2$	1

$$\Rightarrow \int_0^1 u^2\cdot du = \frac{u^3}{3}\Bigg|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

example:

$$\int \underbrace{x e^{-x^2/2}\cdot dx}$$

$$= \int e^{-t}\, dt$$

$t = \frac{x^2}{2}$

$dt = \frac{2x}{2}dt = x\,dt$

$s = -t$

$ds = (-1)dt \Rightarrow (-1)ds = dt$

$$= \int e^s(-1)ds = (-1)e^s + c = -e^{-t} + c = \boxed{-e^{-\frac{x^2}{2}} + c}$$